Math 304 Midterm 2

Name: ____________________

This exam has 11 questions, for a total of 100 points + 10 bonus points.

Please answer each question in the space provided. You need to write **full solutions**. Answers without justification will not be graded. Cross out anything the grader should ignore and circle or box the final answer.

Question	Points	Score
1	12	
2	5	
3	10	
4	10	
5	15	
6	12	
7	10	
8	16	
9	10	
Total:	100	

Question	Bonus Points	Score
Bonus Question 1	5	
Bonus Question 2	5	
Total:	10	

Question 1. (12 pts)

Determine whether each of the following statements is true or false. You do NOT need to explain.

(a) Given two matrices A and B, if B is row equivalent to A, then B and A have the same rank.

> **Solution:** True.

(b) Let v and w be two nonzero vectors in $\mathbb{R}^5$. If v and w are orthogonal to each other, then v and w are linearly independent.

> **Solution:** True

(c) Suppose $L : V \to W$ is a linear transformation between two vector spaces V and W. Then the kernel of L is a subspace of V.

> **Solution:** True.

(d) For any $(m \times n)$ matrix A, we have

$$\dim N(A) + \operatorname{rank}(A) = n$$

> **Solution:** True.

Question 2. (5 pts)

Find the angle between $v = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 2 \end{pmatrix}$ in $\mathbb{R}^4$.

Solution: We denote the angle between v and w by θ.

$$\cos\theta = \frac{< v, w >}{\|v\|\|w\|} = \frac{2}{\sqrt{5}}$$

So

$$\theta = \arccos(\frac{2}{\sqrt{5}})$$

Question 3. (10 pts)

Let V be the subspace of $\mathbb{R}^3$ spanned by $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Describe $V^\perp$ by finding a basis of $V^\perp$.

Solution: $V^\perp$ consists of vectors (a, b, c) which satisfy the following conditions.

$$\begin{cases} (1,0,0) \cdot (a,b,c) = 0 \\ (1,1,0) \cdot (a,b,c) = 0 \end{cases}$$

That is,

$$\begin{cases} a = 0 \\ a + b = 0 \end{cases}$$

So we have

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = s \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

It follows that

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

forms a basis of $V^\perp$.

Question 4. (10 pts)

Suppose a plane H passes through the point $(1, 1, 0)$ and is orthogonal to the line

$$L : x = 2t + 1,\ \ y = 4t - 2,\ \ z = t + 5.$$

Find the equation of H.

Solution: $n = (2, 4, 1)$ is a normal vector of H. So the equation of H is

$$2x + 4y + z = d.$$

Plug the point $(1, 1, 0)$ into the equation to solve for d. We get

$$2x + 4y + z = 6.$$

Question 5. (15 pts)
Given

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 3 & 4 & -6 \end{bmatrix}$$

(a) Find a basis of Ker(A).

Solution: First, use elementary row operations to get a row echelon form of A.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

So all elements in Ker$A$ are of the form

$$t \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

So

$$v = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

forms a basis of the kernel.

(b) Find a basis of the row space of A.

Solution: The two nonzero rows in the row echelon form of A form a basis of the row space of A. That is

$$u_1 = (1, 0, 2)$$

$$u_2 = (0, 1, -3)$$

form a basis of the row space of A.

(c) Find a basis of the range of A.

Solution: Note that the range of A is the same as the column space of A. Use the row echelon from the part (a), we see that the 1st and 2nd columns of A form a basis of the range of A. That is,

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

form a basis of the range of A.

(d) Determine the rank of A.

Solution: The rank of A is the dimension of the row space. So the rank of A is 2. (In fact, we know that the rank of A is also the same as the dimension of the column space of A.)

Question 6. (12 pts)

Determine whether the following mappings are linear transformations.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^1$ by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + x_2 + 1$$

Solution: T is not a linear transformation. For example,

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 3$$

on the other hand,

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 + 2 = 4$$

So

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \neq T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b) Let $M_{n\times n}(\mathbb{R})$ be the vector space of all $(n \times n)$ matrices. Let us fix an $(n \times n)$ matrix A and define a mapping $L : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ by

$$L(B) = AB.$$

That is, for any $(n \times n)$ matrix B, the mapping L maps it to the product of A and B.

Solution: Suppose $B_1, B_2 \in M_{n\times n}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} L(\alpha B_1 + \beta B_2) &= A(\alpha B_1 + \beta B_2) = \alpha AB_1 + \beta AB_2 \\ &= \alpha L(B_1) + \beta L(B_2) \end{aligned}$$

So L is a linear transformation.

Question 7. (10 pts)

Let $M_2(\mathbb{R})$ be the vector space of all (2×2) matrices with real coefficients. The set

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis of $M_2(\mathbb{R})$. Find the coordinates of $A = \begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix}$ with respect to the basis S.

Solution: We need to write

$$\begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix} = a \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

That is, we need to solve the linear system

$$\begin{cases} a + b + c + 2d = 5 \\ a + b + 2c = 3 \\ a + 2b = 3 \\ a = 1 \end{cases}$$

Simply use back substitution. We have

$$a = 1, b = 1, c = 1/2, d = 5/4$$

So

$$[A]_S = \begin{bmatrix} 1 \\ 1 \\ 1/2 \\ 5/4 \end{bmatrix}$$

Question 8. (16 pts)

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 2x_2 \end{pmatrix}$$

(a) Find the matrix representation of L with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

Solution:

$$L(e_1) = L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, L(e_2) = L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

So the matrix representation of L with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$$

(b) Let $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Moreover, let $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Find the matrix representation of L with respect to the basis $\{u_1, u_2\}$ of $\mathbb{R}^2$ and the basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$.

Solution:

$$L(u_1) = L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad L(u_2) = L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$$

Now we need to find the coordinate vectors of $L(u_1)$ nad $L(u_2)$ with respect to the basis $B = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$.

$$L(u_1) = av_1 + bv_2 + cv_3$$

Solve for a, b, c, and we get $[L(u_1)]_B = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Similarly, we obtain

$$[L(u_2)]_B = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

So the matrix representation of L in this case is

$$\begin{pmatrix} 0 & 2 \\ 1 & -2 \\ 0 & 2 \end{pmatrix}$$

Question 9. (10 pts)

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Its matrix representation with respect to the standard basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

(a) Find the transition matrix from the basis $\{u_1, u_2\}$ to the standard basis $\{e_1, e_2\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution: The transition matrix is

$$U = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(b) Find the matrix representation of L with respect to $\{u_1, u_2\}$.

Solution: The matrix representation of L with respect to $\{u_1, u_2\}$ is

$$U^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} U,$$

where U is the matrix from part (a). In this case, an easy way to find U^{-1} is

$$U^{-1} = \frac{1}{\det U} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

So

$$U^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} U = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Bonus Question 1. (5 pts)

Recall that $S = \{1, t, t^2\}$ is a basis of $\mathbb{P}_2(t)$. Let $F : \mathbb{P}_2(t) \to \mathbb{P}_2(t)$ be the linear transformation defined by

$$F(1) = 1 + 2t + t^2, F(t) = 2 + t + t^2 \text{ and } F(t^2) = -1 + t$$

(a) Write down the matrix representation of F relative to the basis $S = \{1, t, t^2\}$.

Solution:

$$[F]_S = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(b) Find the kernel of F.

Solution: First reduce the matrix $[F]_S$ in part (a) to its echelon form, which is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\text{Ker}F = \text{span}\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\}$. In other words, $\text{Ker}F$ is spanned of one polynomial $-1 + t + t^2$.

(c) Find the dimension of the range of F.

Solution:

$$\dim(\text{Im}F) + \dim(\text{Ker}F) = 3$$

From part (b), we know that $\dim(\text{Ker}F) = 1$. So $\dim(\text{Im}F) = 2$.

Bonus Question 2. (5 pts)

Given an $(n \times n)$ matrix $A = (a_{ij})$, we define its *trace*, denoted by $\mathrm{Tr}(A)$, to be

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

(a) It is a fact that for any two $(n \times n)$ matrices A and B, we have

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

Give a proof of this fact in the case when $n = 3$. That is, prove that for any two (3×3) matrices A and B, we have $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Solution: This is done by a direct computation. Let us write $A = (a_{ij})$ and $B = (b_{ij})$. Then (i, j) entry of AB is

$$\sum_{k=1}^{3} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j},$$

and the (i, j) entry of BA is

$$\sum_{k=1}^{3} b_{ik}a_{kj} = b_{i1}a_{1j} + b_{i2}a_{2j} + b_{i3}a_{3j}.$$

Now

$$\begin{aligned}
\mathrm{Tr}(AB) &= \sum_{k=1}^{3} a_{1k}b_{k1} + \sum_{k=1}^{3} a_{2k}b_{k2} + \sum_{k=1}^{3} a_{3k}b_{k3} \\
&= \sum_{k=1}^{3} b_{k1}a_{1k} + \sum_{k=1}^{3} b_{k2}a_{2k} + \sum_{k=1}^{3} b_{k3}a_{3k} \\
&\text{regroup terms} \\
&= \sum_{\ell=1}^{3} b_{1\ell}a_{\ell 1} + \sum_{\ell=1}^{3} b_{2\ell}a_{\ell 2} + \sum_{\ell=1}^{3} b_{3\ell}a_{\ell 3} = \mathrm{Tr}(BA)
\end{aligned}$$

(b) Now use the fact from part (a) to show that if A and B are similar, then

$$\mathrm{Tr}(A) = \mathrm{Tr}(B).$$

Solution: If A and B are similar, then there exist S such that $A = S^{-1}BS$.

$$\mathrm{Tr}(A) = \mathrm{Tr}(S^{-1}BS) = \mathrm{Tr}\big((S^{-1}B)S\big) = \mathrm{Tr}\big(S(S^{-1}B)\big) = \mathrm{Tr}(S^{-1}SB) = \mathrm{Tr}(B)$$